# Solution Sheet 4

**Exercise 1.** (for credit, due on 12 October)
Let $\mu_n = \{\zeta \in \mathbb{C} : \zeta^n = 1\}$ act on $\mathbb{CP}^1$ by $z \mapsto \zeta z$ (with $\infty$ fixed). Let $\pi : \mathbb{CP}^1 \to \mathbb{CP}^1/\mu_n$ be the quotient map.

1. *(2 points)* Show that $\mathbb{CP}^1/\mu_n$ and $\mathbb{CP}^1$ are biholomorphic.
2. *(1 point)* Compute the degree of $\pi$.
3. *(1 point)* Find the ramification points of $\pi$ and their ramification indices.
4. *(1 point)* Find the branch points of $\pi$.

**Solution 1.**

1. Consider the map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ given by
$$f(z) = \begin{cases} z^n, & z \in \mathbb{C}, \\ \infty, & z = \infty. \end{cases}$$
It is holomorphic and $\mu_n$-invariant, so it factors through the quotient $f = F \circ \pi$, with $F : \mathbb{CP}^1/\mu_n \to \mathbb{CP}^1$ holomorphic. The map $F$ is surjective: For $w \in \mathbb{C}^\times$, choose $z$ with $z^n = w$; also $f(0) = 0, f(\infty) = \infty$. So $f$, and hence $F$, is surjective. The map $F$ is injective: Suppose $F(\pi(z)) = F(\pi(z'))$, so $w = f(z) = f(z')$. If $w \in \mathbb{C}^\times$, then $z^n = z'^n$. Thus $z' = \zeta z$ for some $\zeta \in \mu_n$, i.e. $\pi(z) = \pi(z')$. If $w = 0$, then we must have $z = 0 = z'$, and hence $\pi(z) = \pi(z')$; if $w = \infty$, then $z = \infty = z'$. Hence $F$ is a bijective holomorphic map between Riemann surfaces, and thus a biholomorphism.
2. For a generic $y \in \mathbb{CP}^1/\mu_n$ (i.e. $y \notin \{\pi(0), \pi(\infty)\}$), choose $z \in \mathbb{CP}^1$ with $\pi(z) = y$. At such a point $z$, the action of $\mu_n$ has trivial stabilizer. The fiber of $y$ is the $\mu_n$-orbit
$$\pi^{-1}(y) = \{\zeta z : \ \zeta \in \mu_n\},$$
which has $n$ points. Since the stabilizer at $z$ is trivial, the quotient map is locally a biholomorphism, and therefore all points in the fiber $\pi^{-1}(y)$ are unramified. It follows that $\pi$ has degree $n$.
3. Let $\mu_n$ act on a small disk $D = \{|u| < \varepsilon\} \subset \mathbb{C}$ by $u \mapsto \zeta u$. There is a local coordinate $y$ on $D/\mu_n$ near $\pi(0)$ such that $y \circ \pi(u) = u^n$. In particular, in the source coordinate $u$ and the target coordinate $y$, the quotient map is $u \mapsto u^n$, so the ramification index at 0 is $n$. Near $\infty$ we use the coordinate $w = 1/z$. The action is then $w \mapsto \zeta^{-1} w$, so the invariant coordinate is $w^n$. Defining a quotient coordinate $x$ by $x \circ \pi = w^n$ shows that the ramification index at $\infty$ is also $n$.
4. The branch points of $\pi$ are $\pi(0)$ and $\pi(\infty)$.

**Exercise 2.** Let $P(x, y) = x^2 - y^3$ and consider the smooth affine curve
$$C_P := \{(x, y) \in \mathbb{C}^2 : \ P(x, y) = 0, \ (\partial_x P, \partial_y P) \neq (0, 0)\}.$$
Show that
$$\nu : \mathbb{C} \setminus \{0\} \to C_P, \qquad \nu(t) = (t^3, t^2)$$
is a biholomorphism.

**Solution 2.** From $P(t^3, t^2) = t^6 - t^6 = 0$ we see that the point $\nu(t)$ lies on the curve $C_P$. Furthermore, for $t \neq 0$ we have $\partial_x P = 2x = 2t^3 \neq 0$, $\partial_y P = -3y^2 = -3t^4 \neq 0$, so $\nu(t)$ is a smooth point. Thus $\nu$ is well defined and holomorphic (polynomial map). Next, any $(x, y) \in C_P$ satisfies $y \neq 0$: if $y = 0$, then $x^2 = 0 \Rightarrow (x, y) = (0, 0)$, but at $(0, 0)$ the gradient vanishes, so $(0, 0) \notin C_P$. Define
$$\phi : C_P \to \mathbb{C} \setminus \{0\}, \qquad \phi(x, y) = \frac{x}{y}.$$

This is holomorphic on $C_P$ (a rational function with nonvanishing denominator). For $(x, y) \in C_P$, since $x^2 = y^3$ and $y \neq 0$,

$$\phi(x,y)^2 = \frac{x^2}{y^2} = \frac{y^3}{y^2} = y, \qquad \phi(x,y)^3 = \frac{x^3}{y^3} = \frac{x^3}{x^2} = x,$$

so $\nu(\phi(x, y)) = (x, y)$. Conversely, for $t \in \mathbb{C} \setminus \{0\}$, $\phi(\nu(t)) = t^3/t^2 = t$. Hence $\phi = \nu^{-1}$, and $\nu$ is a biholomorphism between $\mathbb{C} \setminus \{0\}$ and $C_P$.

**Exercise 3.** Fix distinct $a_1, \ldots, a_{2g+2} \in \mathbb{C}$ and let

$$P(x,y) = y^2 - \prod_{j=1}^{2g+2} (x - a_j).$$

Consider the hyperelliptic curve of genus $g$

$$C_P := \{(x,y) \in \mathbb{C}^2 : \ P(x,y) = 0, \ (\partial_x P, \partial_y P) \neq (0,0)\}.$$

Let $\pi : C_P \to \mathbb{C}$ be the projection $(x, y) \mapsto x$. Find the ramification points of $\pi$, their multiplicities and compute the degree of $\pi$.

**Solution 3.** Let us write $f(x) = \prod_{j=1}^{2g+2}(x - a_j)$, so $P(x, y) = y^2 - f(x)$. Set

$$U_x := C_P \cap \{\partial_y P \neq 0\}, \quad U_y := C_P \cap \{\partial_x P \neq 0\}.$$

By the implicit function theorem, on the chart $U_x$ the curve is given as a graph $y = g(x)$, so $\pi|_{U_x}$ is a holomorphic local biholomorphism, hence unramified there. Ramification can occur only on $C_P \setminus U_x = \{P = 0, \ \partial_y P = 0\}$. For $P(x, y) = y^2 - f(x)$ we have $\partial_y P = 2y$, so the candidate ramification points are exactly

$$\{(a_j, 0) \ : \ 1 \leq j \leq 2g+2\}.$$

Fix $j$. Near $(a_j, 0) \in U_y$, we can write $f(x) = (x - a_j)h(x)$ with $h(a_j) \neq 0$, so that the curve equation is $y^2 = (x - a_j)h(x)$. We want to solve locally for $x$ as a function of the local parameter $u := y$. The implicit function theorem gives a holomorphic $x(u)$ with $x(0) = a_j$, and symmetry in $u$ gives that $x(u)$ must be even, so we can write

$$x(u) = a_j + cu^2 + \cdots .$$

Hence

$$u^2 = (x - a_j)h(x) = (cu^2 + O(u^4))(h(a_j) + O(u^2)) = c\, h(a_j)u^2 + O(u^4).$$

Comparing the $u^2$ coefficient gives $c = 1/h(a_j)$. Hence

$$x(u) = a_j + \frac{u^2}{h(a_j)} + O(u^4),$$

therefore the ramification index of $(a_j, 0)$ is 2.
Finally, for $x \notin \{a_j\}$ the fiber $\pi^{-1}(x) = \{(x, \pm\sqrt{f(x)})\}$ has two points, so the degree of $\pi$ is 2.

**Exercise 4.** Can you find a nonconstant polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that the corresponding affine algebraic curve $C_P \subset \mathbb{C}^2$ is compact in the Euclidean topology?

**Solution 4.** Suppose $C_P$ is an affine curve given by a nonconstant polynomial

$$P(x,y) = \sum_{k=0}^{d} a_k(x) y^k \in \mathbb{C}[x,y], \quad a_k \in \mathbb{C}[x],$$

and assume $C_P$ is compact. Then the projection $\pi : C_P \to \mathbb{C}$ has compact image, hence

$$\pi(C_P) \subset \{|x| \leq M\} \text{ for some } M.$$

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Fix $|x_0| > M$. The fiber over $x_0$ is then empty, so the one-variable polynomial $y \mapsto P(x_0, y) \in \mathbb{C}[y]$ has no complex root. By the fundamental theorem of algebra, it must be constant and nonzero, so all coefficients

$$a_k(x_0) = 0 \quad \text{for all } k \geq 1.$$

This holds for infinitely many $x_0$, therefore $a_k$ for $k \geq 1$ vanishes on an infinite set, thus $a_k \equiv 0$. Hence $P(x, y) = a_0(x)$ depends only on $x$ and

$$C_P = \{(x, y) : a_0(x) = 0\} = a_0^{-1}(0) \times \mathbb{C}.$$

If $a_0$ were a nonzero constant, then $P$ would be constant (and $C_P = \emptyset$). Therefore by our assumption $a_0$ is nonconstant, and $a_0^{-1}(0)$ is finite. It follows that $C_P$ is unbounded, contradicting compactness. Therefore, apart from the empty set, there are no compact affine algebraic curves.